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Non BPS noncommutative vortices

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Abstract: We construct exact vortex solutions to the equations of motion of the abelian Higgs model defined in non commutative space, analyzing in detail the properties of these solutions beyond the BPS point. We show that our solutions behave as smooth deformations of vortices in ordinary space time except for parity symmetry breaking effects induced by the non commutative parameter θ.

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Contents

1. Introduction

The study of noncommutative solitons and instantons — finite energy or finite action solutions to the classical equations of motion of noncommutative field theories — has been a field of intense activity after the revival of interest in these theories in connection with strings and brane dynamics (see [1]–[3] and references therein). In fact, the first explicit instanton solutions that were constructed in four dimensional Yang-Mills theory [4] strongly influenced developments in string quantization [5]. Concerning solitons, not only the noncommutative counterparts of vortex, monopoles and other localized solutions in ordinary space were constructed but regular stable solutions which become singular in the commutative limit were also discovered [6]–[21].

Most of these solitons correspond to selfdual/anti-selfdual (BPS) solutions which are more simple to obtain than those arising from the Euler Lagrange (EL) equations of motion. Moreover, in even dimensional spaces, calculations can be simplified by exploiting the connection between noncommutative Moyal product in configuration space and a Hilbert space representation which realizes noncommutativity in terms of creation and annihilation operators acting on a Fock space.

Among the BPS soliton solutions that have been obtained in this way, particular interest has attracted the construction of noncommutative BPS vortices — static solutions of the noncommutative version of the abelian Higgs model, both when the gauge field dynamics is governed by Maxwell and/or Chern-Simons actions [8]–[15]. The moduli space of these BPS vortices has been studied in detail [15, 20] showing an interesting phase diagram with a critical point at some value of the dimensionless parameter resulting from the combination of the gauge coupling constant, the scalar expectation value and the noncommutative parameter.

It is the purpose of the present work to investigate solutions to the EL equations of motion of the noncommutative Higgs model. That is, to find, apart the already known BPS

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and non-BPS noncommutative vortices, new non-BPS solutions which are the noncommutative counterpart of the regular vortices originally introduced by Nielsen and Olesen [22] and numerically constructed in [23].

The paper is organized as follows. In section 2 we present the model and establish our conventions. Then, in section 3, we propose an ansatz to solve the equations of motion in Fock space reducing the problem to the solution of a system of two coupled second order recurrence relations. Via the Moyal correspondence the solutions can be also expressed in ordinary space as an expansion in Laguerre polynomials with coefficients that can be computed numerically. We discuss in detail in this section the properties of vortex solutions with positive magnetic flux and compare them with those of the commutative case. In section 4 we present the analogous discussion for negative magnetic flux. Finally, we summarize our results and conclusions in section 4.

2. The noncommutative abelian Higgs model

We start by defining the Moyal product in four dimensional space-time in the form

$$\phi(x) * \chi(x) = \exp\left(\frac{i\theta^{\mu\nu}}{2}\partial^x_\mu \partial^y_\nu\right) \phi(x) * \chi(y)\Big|_{y=x} \tag{2.1}$$

with $\theta^{\mu\nu}$ a real antisymmetric constant matrix. Since we are looking for static solutions we shall take $\theta^{0i} = 0$ $(i, j = 1, 2, 3)$ and bring θ^{ij} into its canonical form so that

$$\theta^{12} = \theta\,, \qquad \theta^{13} = \theta^{23} = 0\,. \tag{2.2}$$

Dynamics of the model is governed by the lagrangian density

$$L = -\frac{1}{4}F_{\mu\nu} * F^{\mu\nu} + \overline{D_\mu\phi} * D^\mu\phi - \frac{\lambda}{4}(\phi\overline{\phi} - \eta^2)^2 \tag{2.3}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i(A_\mu * A_\nu - A_\nu * A_\mu) \tag{2.4}$$

$$D_\mu\phi = \partial_\mu\phi - iA_\mu * \phi\,. \tag{2.5}$$

Here A_μ is a U(1) gauge field and $\phi = \phi^1 + i\phi^2$ a complex scalar. Notice that the gauge coupling constant has been rescaled to 1 and the covariant derivative has been chosen as in the "fundamental" representation. Other cases ("antifundamental" and "adjoint" representations) can be handled in a similar way.

Introducing complex variables

$$z = \frac{1}{\sqrt{2}}(x^1 + ix^2)\,, \qquad \bar{z} = \frac{1}{\sqrt{2}}(x^1 - ix^2) \tag{2.6}$$

the equations of motion read

$$\begin{aligned} D_z D_{\bar{z}}\phi + D_{\bar{z}} D_z\phi &= \frac{\lambda}{2}(\phi\overline{\phi} - \eta^2)\phi \\ D_{\bar{z}} F_{z\bar{z}} &= j_{\bar{z}} \end{aligned} \tag{2.7}$$

where

$$A_z = \frac{1}{\sqrt{2}}(A_1 - iA_2)\,, \qquad A_{\bar z} = \frac{1}{\sqrt{2}}(A_1 + iA_2)$$
$$j_{\bar z} = -i\left((\partial_{\bar z}\phi) * \overline{\phi} - \phi * (\partial_{\bar z}\overline{\phi})\right) - (A_{\bar z} * \phi) * \overline{\phi} - \phi * (\overline{\phi} * A_{\bar z})\,. \tag{2.8}$$

Noncommutative field theories in two dimensional space can be also handled by introducing annihilation and creation operators $\hat a$ and $\hat a^\dagger$ acting on a Fock space,

$$\hat a = \frac{1}{\sqrt{\theta}} z\,, \qquad \hat a^\dagger = \frac{1}{\sqrt{\theta}}\bar z \tag{2.9}$$
$$[\hat a, \hat a^\dagger] = 1 \tag{2.10}$$

in terms of which one takes a field $\phi(z,\bar z)$ as an operator $\hat O_\phi(\hat a, \hat a^\dagger)$. The identity

$$\hat O_\phi(\hat a, \hat a^\dagger)\hat O_\chi(\hat a, \hat a^\dagger) = \hat O_{\phi*\chi}(\hat a, \hat a^\dagger) \tag{2.11}$$

shows that the $*$ product in configuration space becomes the product of operators in Fock space. Moreover, integration in the plane (x^1, x^2) becomes a trace,

$$\int d^2x F(x^1,x^2) = 2\pi\theta\,\mathrm{Tr}\,\hat O_F[\hat a, \hat a^\dagger] \tag{2.12}$$

With the conventions above, derivatives in the Fock space are given by

$$\partial_z = -\frac{1}{\sqrt{\theta}}[\hat a^\dagger,\ ]\,, \qquad \partial_{\bar z} = \frac{1}{\sqrt{\theta}}[\hat a,\ ] \tag{2.13}$$

so that the EL equations of motion (2.7) become the operator equations

$$\frac{1}{\theta}\left([\hat a^\dagger,[\hat a,\hat\phi]] + [\hat a,[\hat a^\dagger,\hat\phi]]\right) - \frac{i}{\sqrt{\theta}}[\hat a^\dagger, \hat A_{\bar z}\hat\phi] + \frac{i}{\sqrt{\theta}}\hat A_z[\hat a,\hat\phi] + \tag{2.14}$$
$$+\left(\hat A_z\hat A_{\bar z} + \hat A_{\bar z}\hat A_z\right)\hat\phi + \frac{i}{\sqrt{\theta}}[\hat a, \hat A_z\hat\phi] - \frac{i}{\sqrt{\theta}}\hat A_{\bar z}[\hat a^\dagger,\hat\phi] = -\frac{\lambda}{2}(\hat\phi\overline{\hat\phi} - \eta^2)\hat\phi$$
$$\frac{1}{\theta}\left([\hat a,[\hat a^\dagger,\hat A_{\bar z}]] + [\hat a,[\hat a,\hat A_z]] + i\sqrt{\theta}[\hat a,[\hat A_z,\hat A_{\bar z}] + i\sqrt{\theta}[\hat A_{\bar z},[\hat A_z,\hat A_{\bar z}]]\right) + \tag{2.15}$$
$$+\frac{i}{\sqrt{\theta}}([\hat A_{\bar z},[\hat a^\dagger,\hat A_{\bar z}]] + [\hat A_{\bar z},[\hat a,\hat A_z]] - \frac{i}{\sqrt{\theta}}\left([\hat a,\hat\phi]\overline{\hat\phi} - \hat\phi[\hat a,\overline{\hat\phi}]\right) = -(A_{\bar z}\hat\phi\overline{\hat\phi} + \hat\phi\overline{\hat\phi}A_{\bar z})$$

When

$$\lambda = \lambda_{\mathrm{BPS}} = 2 \tag{2.16}$$

— the Bogomol'nyi point — solutions of the "BPS" equations

$$B = \eta^2 - \hat\phi\overline{\hat\phi}\,, \qquad D_{\bar z}\hat\phi = 0 \tag{2.17}$$
$$-B = \eta^2 - \hat\phi\overline{\hat\phi}\,, \qquad D_z\hat\phi = 0 \tag{2.18}$$

also solve the Euler-Lagrange equations of motion (2.14)–(2.15) and saturate a lower bound for the energy (eqs. (2.17) and (2.18) correspond to the dual and self-dual cases respectively). Notice that the BPS point corresponds to the case in which the scalar mass m_ϕ and the vector particle mass m_A ratio, given by

$$\frac{m_\phi^2}{m_A^2} = \frac{\lambda}{2}\,, \tag{2.19}$$

is equals to one. In the Ginzburg Landau version of the theory, the above expression, related to the ratio of the condensate coherent length and magnetic penetration length signals the boundary between type-I and type-II superconductors. In the first case, $\lambda < \lambda_{\rm BPS}$ the range of matter self-interaction exceeds that of the electromagnetic one leading to an attractive vortex-vortex interaction while for the second case the opposite is true.

As mentioned above, exact vortex solutions to the selfdual eqs. (2.17) have been constructed for the whole range $0 \le \theta\eta^2 \le \infty$ [12]. They are the counterpart of regular vortex solutions to Bogomol'nyi equations in the commutative case and in fact one can see that they reduce to the exact solutions found in [24] in the $\theta \to 0$ limit. Concerning the antiselfdual case (2.18), it has been shown in [15] that solutions exist only in the range $0 \le \theta\eta^2 \le 1$. At the critical point $\theta\eta^2 = 1$, the BPS solution in this anti-selfdual case coincides with the fluxon solution discovered in [9]. As for non-BPS solutions to the Euler-Lagrange equations (2.7), to our knowledge, the only reported explicit vortex solutions correspond to non-BPS fluxons [11, 15], which exist only in the anti-selfdual case, are unstable in the range $0 \le \theta\eta^2 < 1$ and become singular in the commutative limit.

3. Vortex solutions for positive flux

Vortex configurations in commutative space take the form [22]

$$\phi = f(z\overline{z})\frac{z^M}{(z\overline{z})^{\frac{M}{2}}} \qquad A_z = -iM\frac{d(\bar{z}z)}{z} \tag{3.1}$$

$$\phi = f(z\overline{z})\frac{\bar{z}^M}{(z\overline{z})^{\frac{M}{2}}} \qquad A_z = iM\frac{d(\bar{z}z)}{z} \tag{3.2}$$

for magnetic flux Φ proportional to $+M$ and $-M$ respectively. Inspired in (3.1)–(3.2), we propose the following ansatz in order to construct exact solutions to the equations of motion (2.14)–(2.15) for arbitrary values of the noncommutative parameter θ and $\Phi \ge 1$,

$$\begin{aligned} \hat{\phi} &= \eta\sum_n f_n|n\rangle\langle n+M| \\ \hat{A}_z &= \frac{i}{\sqrt{\theta}}\sum_n (t_n + \sqrt{n+1})|n+1\rangle\langle n| \end{aligned} \tag{3.3}$$

(We leave for the next section the case of negative flux). Plugging the ansatz (3.3) into eqs. (2.15) we get the following recurrence relations for coefficients f_1 and t_1,

$$\begin{aligned} 2\Big(t_n f_{n+1}\sqrt{n+1+M} + t_{n-1}f_{n-1}\sqrt{n+M}\Big) + (t_n^2 + t_{n-1}^2 + 2n + 2M + 1)f_n &= \\ = -\frac{\theta\eta^2\lambda}{2}f_n({f_n}^2 - 1) \end{aligned} \tag{3.4}$$

$$(t_{n+1}^2 - 2t_n^2 + t_{n-1}^2)t_n = \theta\eta^2\left(2f_nf_{n+1}\sqrt{n+1+M} + ({f_n}^2 + {f_{n+1}}^2)t_n\right) \tag{3.5}$$

$$\begin{aligned} f_1 &= -\frac{f_0}{2t_0\sqrt{1+M}}\left((1+2M) + t_0^2 + \frac{\theta\eta^2\lambda}{2}({f_0}^2 - 1)\right) \\ t_1^2 &= 2t_0^2 + \theta\eta^2\left((f_1^2 + f_0^2)t_0 + 2\frac{\sqrt{1+M}f_0f_1}{t_0}\right) \end{aligned} \tag{3.6}$$

Given a value for f_0 and t_0, one can then determine all $f_n's$ and $t_n's$ from eqs. (3.4)–(3.6). The correct values for f_0 and e_0 should make

$$f_n^2 \to 1\,, \quad t_n \to -\sqrt{M+n+1} \qquad \text{as } n \to \infty \tag{3.7}$$

which, as can be seen from ansatz (3.3), correspond to a scalar field going to its v.e.v. and a gauge field going to a pure gauge for $r \to \infty$ (the radial variable r is related to the number operator $\hat{N}$ in Fock space).

Once all $f_n's$ and $t_n's$ are calculated, one can compute all relevant quantities. In particular, the vortex magnetic field can be computed from

$$-i\hat{F}_{z\bar{z}} \equiv \hat{B} = \frac{1}{\theta}\sum_n \mathcal{B}_n |n\rangle\langle n| \tag{3.8}$$

where

$$\hat{B} = \frac{1}{\theta}\sum_n \mathcal{B}_n |n\rangle\langle n| \tag{3.9}$$

and

$$\begin{aligned} \mathcal{B}_0 &= t_0^2 - 1 \\ \mathcal{B}_n &= t_n^2 - t_{n-1}^2 - 1\,, \qquad n \geq 1 \end{aligned} \tag{3.10}$$

One can easily calculate (without the need to use the equations of motion) the magnetic flux Φ

$$\Phi = 2\pi\theta\,\mathrm{Tr}\,\hat{B} = 2\pi M \tag{3.11}$$

Expressions in Fock space can be pulled back to configuration space by using the Moyal mapping. For instance, using the explicit formula for $|n\rangle\langle n|$ in configuration space in terms of Laguerre polynomials L_n one ends with

$$B(r) = \frac{2}{\theta}\sum_n (-1)^n \mathcal{B}_n \exp\left(-\frac{r^2}{\theta}\right) L_n\left(\frac{2r^2}{\theta}\right) \tag{3.12}$$

Using the expression for the energy-momentum tensor,

$$T_{00} = 2\pi\theta\,\mathrm{Tr}\left(\frac{1}{2}\hat{B}^2 + 2\overline{D_z\phi}D_z\phi + 2\overline{D_{\bar{z}}\phi}D_{\bar{z}}\phi + \frac{\lambda}{4}\left(\phi\bar{\phi} - \eta^2\right)^2\right) \tag{3.13}$$

one can write the energy of the vortex configuration in terms of coefficients $f_n's$ and $t_n's$ as

$$\begin{aligned} E^{(M)} = 2\pi\sum_n\Bigg(&\frac{1}{2\theta}(t_n^2 - t_{n-1}^2 - 1)^2 + \eta^2\bigg(\Big(f_n t_n + f_{n+1}\sqrt{n+M+1}\Big)^2 + \\ &+\Big(f_{n+1}t_n + f_n\sqrt{n+M+1}\Big)^2\bigg) + \frac{\lambda\theta\eta^4}{4}({f_n}^2 - 1)^2\Bigg) \end{aligned} \tag{3.14}$$

For simplicity, we shall first discuss the $M = 1$ case and then comment the case of arbitrary positive integer M. Exploring the whole range of $\theta\eta^2$ and λ one finds that vortex solutions exist for all the values of λ and $\theta\eta^2$ considered. For $\lambda_{\rm BPS} = 2$, the

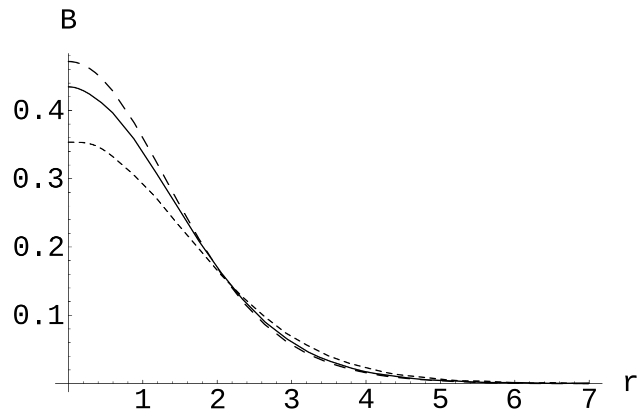

Figure 1: The $M = 1$ vortex magnetic field B as a function of r (in units of η^2) for $\theta\eta^2 = 2$ and different values of λ: the dotted line corresponds to $\lambda = 0.5$, the solid one to $\lambda = 2$ (the BPS point) and the dashed one to $\lambda = 8$.

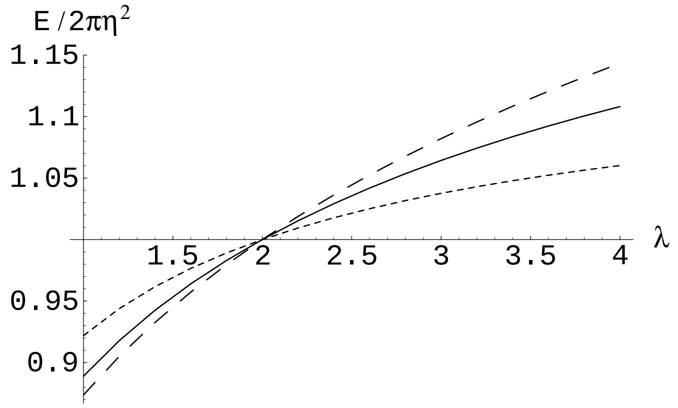

Figure 2: Energy per unit length (in units of η^2) as a function of λ: the dashed line corresponds to $\theta\eta^2 = 0.1$, the solid one to $\theta\eta^2 = 0.5$ and the dotted one to $\theta\eta^2 = 2$.

solution coincides with that obtained in [12] by solving the BPS equations. Concerning the commutative limit (small-θ regime) we reobtain the exact solution found in [24] for $\lambda = \lambda_{\rm BPS}$ as well as the variational results obtained in [23] for $1 \leq \lambda \leq 3$. As an illustration, we show in figure 1 the $M = 1$ vortex magnetic field as a function of θr for $\theta\eta^2 = 2$, and different values of λ. Other ranges of parameters give similar behavior.

In figure 2 we show the energy $E^{(1)}$, as a function of λ, for different values of θ. The energy of all solutions coincide at Bogomol'nyi point ($\lambda = 2$) as already established in [12],

$$\frac{1}{\eta^2} E^{(1)} \left[\lambda = \lambda_{\rm BPS}; \theta\eta^2\right] = 2\pi \tag{3.15}$$

Outside the BPS point the energy is θ dependent and one finds, on the one hand

$$\frac{\delta E^1(\lambda, \theta)}{\delta\theta} > 0 \qquad \lambda < \lambda_{\rm BPS} \tag{3.16}$$

$$\frac{\delta E^1(\lambda, \theta)}{\delta\theta} < 0 \qquad \lambda > \lambda_{\rm BPS} \tag{3.17}$$

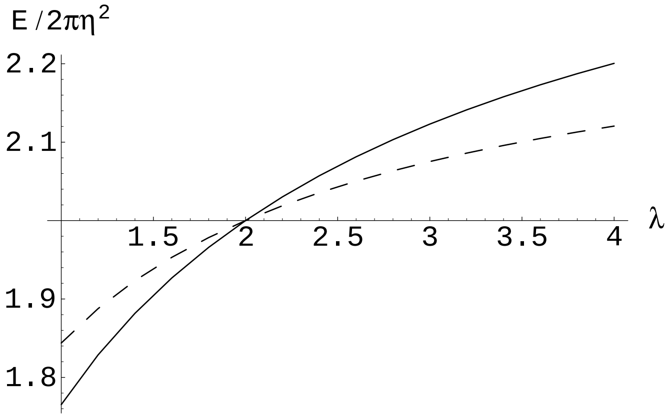

Figure 3: The energy of an $M = 2$ vortex as a function of λ (solid line) compared with that corresponding to twice the energy of an $M = 1$ vortex (dashed line), for $\theta\eta^2 = 2$.

On the other hand, one also has in the whole θ range

$$\frac{\delta E^1(\lambda, \theta)}{\delta\lambda} > 0 \tag{3.18}$$

The calculations described above can be easily extended to the search of vortex solutions with arbitrary positive flux M. The resulting field configurations for $M > 1$ are qualitatively similar to the $M = 1$ case.

Nevertheless, it is important in this case, to compare the energy of the M-vortex $E^{(M)}$ with $E^{(1)}$. We show in figure 3 the energy of a $M = 2$ vortex compared with twice the energy for an $M = 1$ vortex as a function of λ for fixed θ ($\theta\eta^2 = 2$). In complete analogy with the commutative case, there is a crossover at the Bogomol'nyi point $\lambda_{\rm BPS}$ signaling that for $\lambda > \lambda_{\rm BPS}$ it is energetically favorable for a $M = 2$-vortex configuration to decay into two $M = 1$ vortices. This behavior indicates that, as in the commutative case, vortices attract (repel) each other for values of the coupling constant below (above) the Bogomol'nyi point. This behavior remains the same for all values of $\theta\eta^2$ investigated indicating that the character of attraction/repulsion is unaffected by the value of the parameter $\theta\eta^2$. Of course, at $\lambda = \lambda_{\rm BPS}$ vortices do not interact (the stress tensor vanishes [24]). In this case one has $E^{(2)}[\lambda_{\rm BPS}] = 2E^{(1)}[\lambda_{\rm BPS}]$.

Let us end this section with a comment about the accuracy of our numerical computations. When solving the recursive relations that define the solutions for the coefficients f_n and t_n, we have truncated the Fock space to a given value of n. Since the recursive relations are highly nonlinear, it is very difficult to have a controlled management of the errors due to that truncation. However, we can have an estimate of the error in the computation of the energy by comparing the numerical result at the Bogol'nyi point $\lambda = \lambda_{\rm BPS}$ (for $M = 1, 2$, for example) with the exact analytical results. We found that for the range of values of $\theta\eta^2$ considered, the error is less than 10^{-5}. Our numerical analysis suggests that this estimate of the error can be extrapolated to the values of λ considered in the article.

4. Vortex solutions for negative flux

Since the noncommutativity of space breaks the parity invariance of the theory, negative flux solutions cannot be obtained from the positive flux ones by a parity transformation, as in the commutative case. Negative flux solutions have then to be studied separately. Thus, instead of ansatz (3.3) one has to look, in the case of negative magnetic flux, for configurations in the form

$$\hat{\phi} = \eta \sum_n f_n |n + M\rangle\langle n| \tag{4.1}$$

$$\hat{A}_z = \frac{i}{\sqrt{\theta}} \sum_n (t_n + \sqrt{n+1})|n+1\rangle\langle n| \tag{4.2}$$

$$f_n^2 \to 1\,, \quad t_n \to -\sqrt{M+n-1} \qquad \text{as } n \to \infty \tag{4.3}$$

where M is again a positive integer, $M > 0$, this leading to a negative magnetic flux $\Phi/(2\pi) = -M$.

For simplicity, we present in detail the case $M = 1$ but the analysis goes the same for arbitrary M. Using (4.2), the equations of motion (2.15) lead to the recurrence relations for $n > 1$

$$\begin{aligned} 2\left(t_{n+1} f_{n+1}\sqrt{n+1} + t_n f_{n-1}\sqrt{n}\right) + \left(t_{n+1}^2 + t_n^2 + 2n + 1\right) f_n = \\ = -\frac{\theta\eta^2\lambda}{2} f_n ({f_n}^2 - 1) \end{aligned} \tag{4.4}$$

$$(t_{n+1}^2 - 2t_n^2 + t_{n-1}^2)t_n = \theta\eta^2\left(2f_n f_{n-1}\sqrt{n} + ({f_n}^2 + {f_{n-1}}^2)t_n\right) \tag{4.5}$$

$$f_1 = -\frac{f_0}{2t_1}\left(1 + t_0^2 + t_1^2 + \frac{\theta\eta^2\lambda}{2}({f_0}^2 - 1)\right)$$

$$t_1 = \sqrt{2t_0^2 + \theta\eta^2 f_0^2} \tag{4.6}$$

Again, once all $f_n's$ and $t_n's$ are calculated, one can compute the vortex magnetic field, magnetic flux and energy (since the ansatz for the gauge field is the same as in the positive flux case, the magnetic field is again given by eqs. (3.9), (3.10).

The expression for the energy for a $\Phi/(2\pi) = -1$ configuration takes the form

$$\begin{aligned} E^{(M)} = 2\pi \sum_n \Bigg(\frac{1}{2\theta}\left(t_n^2 - t_{n-1}^2 - 1\right)^2 + \eta^2\Big(\left(f_n t_{n+1} + f_{n+1}\sqrt{n+1}\right)^2 + \\ + \left(f_n t_n + f_{n-1}\sqrt{n}\right)^2\Big) + \frac{\lambda\theta\eta^4}{4}\left({f_{n-1}}^2 - 1\right)^2 \Bigg) \end{aligned} \tag{4.7}$$

(the summation goes from $n = 0$ to $n = \infty$ with the proviso that coefficients with negative subindex vanish).

As shown in [9, 11] and [15], there exist in this case a solution with magnetic flux $\Phi/2\pi = -1$ (a "fluxon") of the form,

$$\begin{aligned} \phi^{fl} &= \eta \sum_{n=0} |n+1\rangle\langle n| \\ A_z^{fl} &= \frac{i}{\sqrt{2\theta}} \sum_{n=0} \left(\sqrt{n+1} - \sqrt{n}\right) |n+1\rangle\langle n| \end{aligned} \tag{4.8}$$

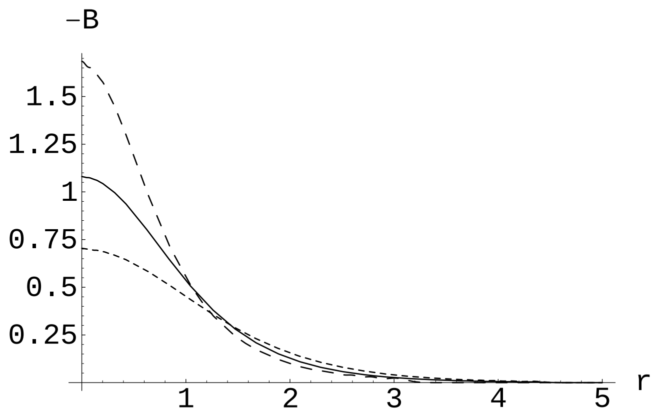

Figure 4: The magnetic field as a function of r for $\theta\eta^2 = 0.1$ and different values of λ: the dotted line corresponds to $\lambda = 0.5$, the solid one to $\lambda = 2$ and the dashed one to $\lambda = 8$.

Indeed, within this ansatz

$$\begin{aligned} -B^{fl} &= \frac{1}{\theta}|0\rangle\langle 0| \\ D_z\phi^{fl} &= D_{\bar{z}}\phi^{fl} = 0 \\ \eta^2 - \phi^{fl}\bar{\phi}^{fl} &= \eta^2|0\rangle\langle 0| \end{aligned} \tag{4.9}$$

By direct substitution, it is then immediate to show that this configuration satisfies the EL equations of motion for all value of the parameters. The energy of the fluxon solution (4.8) is

$$\frac{E^{fl}}{2\pi\eta^2} = \frac{1}{2}\left(\frac{1}{\theta\eta^2} + \frac{\lambda}{2}\theta\eta^2\right) \tag{4.10}$$

Nevertheless, a more careful study reveals that this solutions are locally stable only for $\theta\eta^2 > 1$. Moreover, they are BPS saturated only when $\lambda = 2$ and $\theta\eta^2 = 1$ (Note that for $\theta\eta^2 = 1$ and $\lambda = \lambda_{\mathrm{BPS}} = 2$ the energy does correspond to the BPS bound, $E^{fl} = E_{\mathrm{BPS}} = 2\pi\eta^2$).

Since BPS solutions still can be found for $\lambda = 2$ and $\theta\eta^2 < 1$ by considering an ansatz of the form (4.2) and solving the BPS equation, the question that arises concerns the existence and properties of non BPS solutions for $\theta\eta^2 < 1$. In order to answer this question we have investigated the numerical solutions to the recurrence relations in different ranges of $\theta\eta^2$ and λ.

As an illustration, we show in figure 4 the magnetic field as a function of r for $\theta\eta^2 = 0.1$ and different values of λ. We have calculated numerically the magnetic flux for this configuration confirming that it corresponds to one unit of flux.

One can compare the values for the energy given in the table 1 with those resulting from formula (4.10) for fluxons to conclude that the energy of the solutions we have presented in the range $0 \leq \theta\eta^2 < 1$ is lower than that of the (unstable) non-BPS fluxon. Moreover, the energy of our vortex solution tends to the value of the fluxon solution energy for $\theta\eta^2 \to 1$. We show in figure 5 the behavior of f_n as $\theta\eta^2 \to 1$. Indeed at $\theta = 1$ all $f_n's$ and $t_n's$ for our solutions coincide with those of the fluxon solutions which, from that critical value of θ on remain as the only non-trivial solutions.

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$\theta\eta^2$	$E[\lambda = 0.5]$	$E[\lambda = 2]$	$E[\lambda = 8]$
0.1	0.751	1.000	1.400
0.3	0.735	1.000	1.572
0.5	0.716	1.000	1.801
0.8	0.675	1.000	2.206
0.9	0.654	1.000	2.352

Table 1: The energy of vortex with magnetic flux $\Phi/2\pi = -1$ for different values of $\theta\eta^2$ and λ.

$\theta\eta^2$	$E_{\Phi=-1}$	$E_{\Phi=1}$
0.1	0.755	0.762
0.3	0.735	0.775
0.5	0.716	0.785

Table 2: Vortex and anti-vortex energies (in units of $2\pi\eta^2$) for $\lambda = 0.5$.

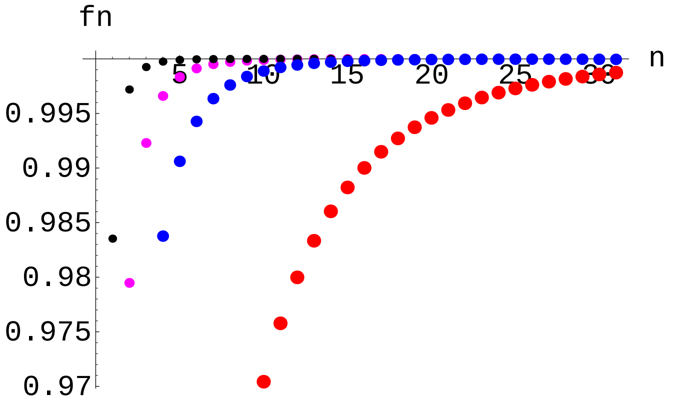

Figure 5: The coefficients f_n in the Higgs field development for the $\Phi/2\pi = -1$ solutions (4.1) as a function of n for different values of θ. Size of dots decreases as $\theta\eta^2$ goes from $\theta\eta^2 = 0.1$ to $\theta\eta^2 = 0.9$.

It is interesting also to notice that the asymmetry between vortex and anti-vortex configurations manifests in the energy splitting between vortex-antivortex configurations for non zero values of $\theta\eta^2$ as shown in table 2. Moreover, the behavior of the energy with θ is the opposite: for the vortex the energy increases (decreases) with θ if $\lambda < 2$ ($\lambda > 2$) while for the antivortex the energy decreases (increases) with θ if $\lambda < 2$ ($\lambda > 2$).

5. Summary and discussion

In this paper we have examined vortex solutions in the abelian Higgs model in non-commutative space, focussing on the properties of these solutions beyond the BPS point previously considered in [8, 11, 12, 15].

Previous to our investigations, the only known non-BPS solutions were fluxons [9, 11], negative flux solutions which are stable only for $\theta\eta^2 > 1$. These configurations, even though they are non-BPS in the sense that they do not satisfy the duality equations, share some properties with BPS solutions, namely, their energy saturates a topological bound and is linear in the flux. Moreover, in the $\theta \to 0$ commutative limite they correspond to singular configurations (with a δ-function source).

We have constructed here non-BPS solutions of positive flux with arbitrary values $\theta\eta^2$ and also negative flux solutions, in this last case in the range $0 \leq \theta\eta^2 < 1$. Unlike the fluxon case mentioned above, no simple analytical expressions of these solutions are available. One has instead expressions like eq. (3.12) so that the properties of the solutions have to be investigated numerically (as it happens in the commutative case, both for BPS and non-BPS solutions [24]–[23]).

The solutions presented here behave in most ways as smooth deformations of vortices in commutative space. For instance, their energy is an increasing function of λ and is a linear function of the flux only at the BPS point. Indeed, we have shown that $E^{(M)} - ME^{(1)} > 0$ for $\lambda > \lambda_{\rm BPS}$ suggesting that in this case, the M-vortex configuration should be unstable towards the formation of a Abrikosov-type vortex lattice in analogy with type-II superconductors. Notice though that solutions in non-commutative space differ from solutions in ordinary space time as a result of parity breaking which manifests itself as a breaking of symmetry between vortex and anti-vortex configurations. We have illustrated this fact by comparing the energies of $E^{(1)}$ and $E^{(-1)}$ as a function of θ.

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